Final Exam — Analysis (WPMA14004)

Tuesday 19 June 2018, 9.00h–12.00h

University of Groningen

Instructions

1. The use of calculators, books, or notes is not allowed.
2. Provide clear arguments for all your answers: only answering "yes", "no", or "42" is not sufficient. You may use all theorems and statements in the book, but you should clearly indicate which of them you are using.
3. The total score for all questions equals 90. If p is the number of marks then the exam grade is $G = 1 + p/10$.

Problem 1 (5 + 10 = 15 points)

(a) Give the definition of a least upper bound.

(b) Assume that the sets $A, B \subset \mathbb{R}$ are non-empty and bounded above. Define

$$A + B = \{a + b \,:\, a \in A \text{ and } b \in B\}.$$

Prove that

$$\sup(A + B) = \sup A + \sup B.$$

Problem 2 (5 + 5 + 5 = 15 points)

Give an example of each of the following, or argue that such a request is impossible:

(a) A sequence that contains subsequences converging to every point in the infinite set $\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\}$;

(b) A convergent series $\sum_{n=1}^{\infty} x_n$ and a bounded sequence (y_n) such that the series $\sum_{n=1}^{\infty} x_n y_n$ diverges;

(c) A sequence (x_n) satisfying $0 \leq x_n \leq 1/n$ for all $n \in \mathbb{N}$ such that the series $\sum_{n=1}^{\infty} (-1)^n x_n$ diverges.

Problem 3 (5 + 5 + 5 = 15 points)

Consider the following set:

$$A = \left\{ -\frac{1}{2}, \frac{3}{4}, -\frac{7}{8}, \frac{15}{16}, -\frac{31}{32}, \frac{63}{64}, -\frac{127}{128}, \ldots \right\}.$$

Show that A is *not* compact in the following ways:

(a) A does not satisfy the definition of a compact set;

(b) A is not closed;

(c) A has an open cover without a finite subcover.

Problem 4 (4 + 7 + 4 = 15 points)

(a) Give the definition of a uniformly continuous function.

(b) Prove that $f(x) = 1/x^2$ is uniformly continuous on $[1, \infty)$.

(c) Is $f(x) = 1/x^2$ also uniformly continuous on $(0, 1]$?

Problem 5 (3 + 6 + 6 = 15 points)

Consider the following sequence of functions:

$$f_n(x) = \frac{n}{nx + 1}.$$

(a) Compute the pointwise limit for all $x \in (0, \infty)$.

(b) Let $a > 0$. Prove that the convergence is uniform on the interval $[a, \infty)$.

(c) Is the convergence also uniform on $(0, \infty)$?

Problem 6 (3 + 8 + 4 = 15 points)

Let $\{r_1, r_2, r_3, \dots\}$ be an enumeration of all rational numbers in $[0, 1]$, and define

$$f(x) = \begin{cases} 1/p^2 & \text{if } x = r_p \text{ for some } p \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Show that $L(f, P) = 0$ for any partition P of $[0, 1]$.

(b) Explain that for any equispaced partition P of $[0, 1]$ with $2n$ subintervals we have

$$U(f, P) \leq \frac{1}{n} \sum_{k=1}^{n} \frac{1}{k^2}.$$

Hint: in the worst case scenario the points $r_1, \dots, r_n$ lie at the boundary of two adjacent subintervals.

(c) Prove that f is integrable on $[0, 1]$ and compute $\int_0^1 f$.

End of test (90 points)

Solution of problem 1 (5 + 10 = 15 points)

(a) A number $s \in \mathbb{R}$ is called a least upper bound for a set $A \subset \mathbb{R}$ if

  (i) $a \leq s$ for all $a \in A$;
  (2 points)

  (ii) if u is an upper bound for A, then $s \leq u$.
  (3 points)

  Equivalent (but not a definition!):
  for any $\epsilon > 0$ there exists $a \in A$ such that $s - \epsilon < a$.
  (2 points)

(b) Write $s = \sup A$ and $t = \sup B$. If $a \in A$ and $b \in B$, then $a \leq s$ and $b \leq t$ implies that $a + b \leq s + t$, which shows that $s + t$ is an upper bound for $A + B$.
(4 points)

In order to show that $s + t$ is the least upper bound of $A + B$ we can proceed in two directions.

Method 1. Let u be any upper bound of $A+B$. Let $b \in B$ be arbitrary, then $a+b \leq u$, or, equivalently, $a \leq u - b$ for all $a \in A$. This shows that $u - b$ is an upper bound of A. Therefore, $s \leq u - b$ since s is the least upper bound of A.
(2 points)

Since b was arbitrary, it follows that $b \leq u - s$ for all $b \in B$. This shows that $u - s$ is an upper bound of B. Therefore, $t \leq u - s$ since t is the least upper bound of B.
(2 points)

Rewriting the inequality gives $s + t \leq u$ which shows that $s + t$ is the least upper bound of $A + B$.
(2 points)

Method 2. For any $\epsilon > 0$ there exist elements $a \in A$ and $b \in B$ such that

$$s - \tfrac{1}{2}\epsilon < a \quad \text{and} \quad t - \tfrac{1}{2}\epsilon < b.$$

(3 points)

This implies that $s + t - \epsilon < a + b$, which means that $s + t - \epsilon$ is not an upper bound of $A + B$. Hence, $s + t$ is the least upper bound of $A + B$.
(3 points)

Note: it is of course also correct to start with $s - \epsilon < a$ and $t - \epsilon < b$ to get $s + t - 2\epsilon < a + b$.

Solution of problem 2 (5 + 5 + 5 = 15 points)

(a) The easiest way to construct such an example is by ensuring that each number of the set $\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots\}$ appears in the sequence an infinite number of times. In this way we can always find a *constant* (and hence convergent) subsequence converging to the desired limits. An example of such a sequence is:

$$\left(1, 1, \frac{1}{2}, 1, \frac{1}{2}, \frac{1}{4}, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots\right).$$

(5 points)

(b) The series $\sum_{n=1}^{\infty}(-1)^{n+1}/n$ converges because it satisfies the conditions of the Alternating Series Theorem. The sequence $y_n = (-1)^{n+1}$ is trivially bounded. However, the series $\sum_{n=1}^{\infty} x_n y_n = \sum_{n=1}^{\infty} 1/n$ diverges.
(5 points)

(c) Take, for example, the sequence

$$x_n = \begin{cases} 0 & \text{if } n \text{ is odd}, \\ 1/n & \text{if } n \text{ is even}. \end{cases}$$

Clearly, $0 \leq x_n \leq 1/n$ for all $n \in \mathbb{N}$. The series $\sum_{n=1}^{\infty}(-1)^n x_n = \frac{1}{2}\sum_{n=1}^{\infty} 1/n$ diverges.
(5 points)

Solution of problem 3 (5 + 5 + 5 = 15 points)

Note that we can write

$$A = \{a_n : n \in \mathbb{N}\} \quad \text{where} \quad a_n = \frac{(-1)^n(2^n - 1)}{2^n}.$$

(a) By definition A is compact if any sequence in A has a convergent subsequence with a limit in A. Note that A is bounded. Indeed,

$$|a_n| = \frac{2^n - 1}{2^n} < 1 \quad \text{for all } n \in \mathbb{N},$$

which shows that $A \in (-1, 1)$. The Bolzano-Weierstrass theorem implies that every sequence in A has a convergent subsequence. However, the limit of such a subsequence does *not* have to be an element of A.

Indeed, the sequence $x_n = a_{2n} = 1 - 1/2^{2n}$ is a sequence in A that already itself converges to 1. Therefore, any subsequence of (x_n) also converges to 1, but $1 \notin A$.
(5 points)

Alternative argument. The sequence $x_n = a_{2n+1} = 1/2^{2n+1} - 1$ is a sequence in A that already itself converges to -1. Therefore, any subsequence of (x_n) also converges to -1, but $-1 \notin A$.

(b) The points 1 and -1 are limit points of A which are not contained in A itself. This follows from a similar reasoning as in part (a). Hence, A is not closed.
(5 points)

(c) Take, for instance, the open intervals $O_n = (-1, a_{2n})$. Then

$$\bigcup_{n=1}^{\infty} O_n = (-1, 1) \supset A,$$

which shows that the collection $\{O_n : n \in \mathbb{N}\}$ forms an open cover for A.
(2 points)

However, finitely many sets O_n do not cover A. Indeed, for natural numbers $n_1 < n_2 < \dots n_k$ we have

$$O_{n_1} \cup O_{n_2} \cup \cdots \cup O_{n_k} = \left(-1, \frac{2^{2n_k} - 1}{2^{2n_k}}\right),$$

which does not contain the points $a_{2n} = (2^{2n} - 1)/2^{2n} \in A$ with $n > n_k$.
(3 points)

Note. Many different covers are possible. For example, one could also take

$$O_n = \left(-\frac{n+1}{n}, \frac{n+1}{n}\right).$$

Solution of problem 4 (4 + 7 + 4 = 15 points)

(a) A function $f : A \to \mathbb{R}$ is uniformly continuous on A if for all $\epsilon > 0$ there exists $\delta > 0$ such that

$$|x - y| < \delta \quad \Rightarrow \quad |f(x) - f(y)| < \epsilon \quad \forall\, x, y \in A.$$

(4 points)

(b) *Method 1.* We have that

$$|f(x) - f(y)| = \left|\frac{1}{x^2} - \frac{1}{y^2}\right| = \left|\frac{y^2 - x^2}{x^2y^2}\right| = \left|\frac{y + x}{x^2y^2}\right| \cdot |x - y|.$$

For all $x, y \in [1, \infty)$ we have

$$\left|\frac{y + x}{x^2y^2}\right| = \frac{y + x}{x^2y^2} = \frac{1}{x^2y} + \frac{1}{xy^2} \leq 1 + 1 = 2,$$

which implies that

$$|f(x) - f(y)| \leq 2|x - y|.$$

(4 points)

Method 2. The function $f(x) = 1/x^2$ is differentiable for all $x \neq 0$. Let $x, y \in [1, \infty)$ and assume $x < y$. By the Mean Value Theorem there exists $c \in (x, y)$ such that

$$f(x) - f(y) = f'(c)(x - y) = -\frac{2}{c^3}(x - y).$$

Hence, if $x, y \geq 1$, then

$$|f(x) - f(y)| = \frac{2}{c^3}|x - y| \leq 2|x - y|.$$

(4 points)

Conclusion. Let $\epsilon > 0$ be arbitrary and set $\delta = \frac{1}{2}\epsilon$. Then

$$|x - y| < \delta \quad \Rightarrow \quad |f(x) - f(y)| \leq 2|x - y| < 2\delta = \epsilon \quad \forall\, x, y \in [1, \infty).$$

(3 points)

(c) Take sequences $x_n = 1/\sqrt{n}$ and $y_n = 1/\sqrt{n+1}$, then $x_n - y_n \to 0$, but $|f(x_n) - f(y_n)| = 1$ for all $n \in \mathbb{N}$. Therefore, f is not uniformly continuous on $(0, 1]$.
(4 points)

Solution of problem 5 (3 + 6 + 6 = 15 points)

(a) Let $x \in (0, \infty)$ be fixed. By the Algebraic Limit Theorem it follows that

$$\lim f_n(x) = \lim \frac{n}{nx+1} = \lim \frac{1}{x+1/n} = \frac{1}{\lim(x+1/n)} = \frac{1}{x+\lim 1/n} = \frac{1}{x}.$$

(3 points)

(b) We have

$$|f_n(x) - f(x)| = \left| \frac{n}{nx+1} - \frac{1}{x} \right| = \left| \frac{nx}{x(nx+1)} - \frac{nx+1}{x(nx+1)} \right| = \frac{1}{x(nx+1)}.$$

Argument 1. Let $a > 0$ be fixed. If $x \in [a, \infty)$, then $x(nx+1) \geq a(na+1) > na^2$ so that

$$|f_n(x) - f(x)| < \frac{1}{na^2} \quad \forall x \in [a, \infty).$$

For $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $1/N < a^2\epsilon$. Hence,

$$n \geq N \quad \Rightarrow \quad |f_n(x) - f(x)| < \frac{1}{na^2} \leq \frac{1}{Na^2} < \epsilon \quad \forall x \in [a, \infty).$$

This shows that $f_n \to f$ uniformly on $[a, \infty)$.
(6 points)

Argument 2. Let $a > 0$ be fixed. We have

$$\sup_{x \in [a,\infty)} |f_n(x) - f(x)| = \frac{1}{a(na+1)} < \frac{1}{na^2},$$

which implies that

$$\lim_{n\to\infty} \left(\sup_{x \in [a,\infty)} |f_n(x) - f(x)| \right) = 0.$$

This shows that $f_n \to f$ uniformly on $[a, \infty)$.
(6 points)

(c) *Argument 1.* In order to satisfy

$$n \geq N \quad \Rightarrow \quad |f_n(x) - f(x)| = \frac{1}{x(nx+1)} < \epsilon$$

we must take $N > (\epsilon - x)/x^2$. This shows that N also depends on x: taking x closer to 0 implies that N has to become larger. Hence, f_n does not converge uniformly to f on $(0, \infty)$.
(6 points)

Argument 2. The function $|f_n(x) - f(x)|$ is unbounded on $(0, \infty)$:

$$\sup_{x \in (0,\infty)} |f_n(x) - f(x)| = \infty.$$

This shows that f_n does not converge uniformly to f on $(0, \infty)$.
(6 points)

Solution of problem 6 (3 + 8 + 4 = 15 points)

(a) Let $P = \{x_0 < x_1 < \dots x_n\}$ be any partition of $[0, 1]$. Every subinterval $[x_{k-1}, x_k]$ contains an irrational number. Therefore,

$$m_k = \inf_{x \in [x_{k-1}, x_k]} f(x) = 0,$$

which implies that

$$L(f, P) = \sum_{k=1}^{n} m_k(x_k - x_{k-1}) = 0.$$

(3 points)

(b) First assume that $P = \{x_0 < x_1 < x_2\}$ is an equispaced partition with 2 intervals. So $x_0 = 0$, $x_1 = \frac{1}{2}$, and $x_2 = 1$. In order to get the largest possible upper sum we have to make the supremum over each subinterval as large as possible. In the worst possible case we have

$$M_1 = \sup_{x \in [x_0, x_1]} f(x) = f(r_1) = 1 \quad \text{and} \quad M_2 = \sup_{x \in [x_1, x_2]} f(x) = f(r_1) = 1.$$

This is the case when $x_1 = r_1$, so when r_1 lies at the common boundary points of the intervals $[x_0, x_1]$ and $[x_1, x_2]$. In this case we have

$$U(f, P) = \sum_{k=1}^{2} M_k(x_k - x_{k-1}) = \sum_{k=1}^{2} \frac{1}{2} = 1.$$

Now let $P = \{x_0 < x_1 < \cdots < x_{2n}\}$ be an equispaced partition of $[0, 1]$ with $2n$ subintervals. By definition we have

$$U(f, P) = \sum_{k=1}^{2n} M_k(x_k - x_{k-1}) = \frac{1}{2n} \sum_{k=1}^{2n} M_k \quad \text{where} \quad M_k = \sup_{x \in [x_{k-1}, x_k]} f(x).$$

In the worst case scenario the points $r_1, \dots, r_n$ lie at the boundary of two adjacent subintervals. This happens, for example, when

$$x_1 = r_1, \quad x_3 = r_2, \quad x_5 = r_3, \quad \dots$$

Note that different orderings are possible.
(4 points for any decent explanation)

Therefore,

$$U(f, P) = \frac{1}{2n} \sum_{k=1}^{2n} M_k \leq \frac{1}{2n} \sum_{k=1}^{n} 2f(r_k) = \frac{1}{n} \sum_{k=1}^{n} \frac{1}{k^2}.$$

(4 points for estimate)

(c) Recall that the series $\sum_{k=1}^{\infty} 1/k^2$ converges. In particular, there exists $C > 0$ such that $\sum_{k=1}^{n} 1/k^2 \leq C$. For any $\epsilon > 0$ there exists $n \in \mathbb{N}$ such that $C/n < \epsilon$. Let $P = \{x_0 < x_1 < \cdots < x_{2n}\}$ be an equispaced partition of $[0, 1]$ with $2n$ subintervals. Then by parts (a) and (b) we have

$$U(f, P) - L(f, P) = U(f, P) \leq \frac{C}{n} < \epsilon,$$

which proves that f is integrable on $[0, 1]$.
(3 points)

Since $L(f, P) = 0$ for all partitions P of $[0, 1]$ it follows that

$$\int_0^1 f = L(f) = \sup\{L(f, P) : P \text{ is a partition of } [0, 1]\} = 0.$$

(1 point)

Alternative argument. By part (b) it follows that

$$\int_0^1 f = U(f) = \inf\{U(f, P) : P \text{ is a partition of } [0, 1]\} = 0.$$

(1 point)